

Compression of quantum measurement operations

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I. INTRODUCTION

We generalize recent work of Massar and Popescu dealing with the amount of classical data that is produced by a quantum measurement on a quantum state ensemble. In the previous work it was shown that quantum measurements generally contain spurious randomness in the outcomes and that this spurious randomness can be eliminated by carrying out collective measurements on many independent copies of the system. In particular it was shown that, without decreasing the amount of knowledge the measurement provides about the quantum state, one can always reduce the amount of data produced by the measurement to the entropy $H(\rho) = -\text{Tr } \rho \log \rho$ of the ensemble.

Here we extend this result by giving a more refined description of what constitute equivalent measurements (that is measurements which provide the same knowledge about the quantum state) and also by considering incomplete measurements. In particular we show that one can always associate to a POVM with elements a_j , an equivalent POVM acting on many independent copies of the system which produces an amount of data asymptotically equal to the entropy defect of an ensemble canonically associated to the ensemble average state ρ and the initial measurement (a_j). In the case where the measurement is not maximally refined this amount of data is strictly less than the amount $H(\rho)$ obtained in the previous work. This result is obtained by a novel technique to analyze random selections. We also show that this is the best achievable, i.e. it is impossible to devise a measurement equivalent to the initial measurement (a_j) that produces less data.

We discuss the interpretation of these results. In particular we show how they can be used to provide a precise and model independent measure of the amount of knowledge that is obtained about a quantum state by a quantum measurement. We also discuss in detail the relation between our results and Holevo's bound, at the same time providing a new proof of this fundamental inequality.

An essential aspect of quantum mechanics is the measurement process. Only by measuring can a macroscopic observer obtain knowledge about a quantum system. However the knowledge that is obtained about the state of a quantum system is in general not complete since from the outcome of a measurement it is in general not possible to infer the initial state. Furthermore there are many different measurements that could be carried out on the system and these measurements are in general mutually incompatible.

It is therefore natural to try to make measurements as efficient as possible. The simplest way one can make a measurement efficient is to devise it in such a way that it provides as much knowledge as possible about the state of the system¹. This first approach has been extensively studied, see for instance [1,2]. We note that in some cases it can be interesting to make an incomplete measurement (which does not provide maximum knowledge about the system). The incomplete measurement can then be refined at a later stage by carrying out a second measurement on the system.

The second way one can make a measurement efficient is to reduce the amount of classical data it produces. Indeed if a measurement produces outcome j with probability p_j , the amount of classical data produced by the measurement is $I = -\sum_j p_j \log p_j$ (in this paper \log and \exp are always to base 2). This second aspect of optimizing measurements was first considered in [3].

Minimizing I is interesting for two reasons. First it makes the measurement less wasteful of resources since it

¹In this article we shall distinguish between the words “knowledge” and “information”. Thus we shall say that a measurement provides knowledge about the state of a quantum system, rather than information. We introduce this distinction because the second term is often associated with the “mutual information” between the initial state and the result of the measurement. And, as examples show, an efficient measurement is not necessarily one that maximizes the mutual information between the initial state and the result of the measurement.

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minimizes that amount of classical data that is produced. Indeed the increase in entropy — in the thermodynamic sense — due to the irreversibility of the measurement process will be minimized if the amount of data I produced by the measurement is minimized. Secondly, as argued in [3], the minimum value of I provides a model independent answer to the question *how much knowledge about a quantum system is obtained by a measurement?*

The main result of [3] was to show that it always possible to reduce I so that it is less or equal than the von Neumann entropy of the ensemble of quantum states on which the measurement is carried out. Thus the answer to the above question is that a quantum measurement can provide at most one bit of classical knowledge about an unknown qubit.

However minimizing I is not an easy task. It must be carried out at the level of the measurement itself and cannot be realized as a post-processing of the data produced by the measurement. This is because there are positive operator valued measures (POVM) that provide maximum knowledge about the state and that have a number of outcomes that is larger than the von Neumann entropy of the ensemble. Such measurements add spurious randomness to their outcomes. To address this difficulty and remove the spurious randomness one must define a notion of “equivalent” measurements that yield the same knowledge about the quantum system and then search among this class of equivalent measurements for those which minimize the number of bits I of classical data produced by the measurement. It is important to include in the equivalence classes not only measurements on single states, but also measurements that act collectively on blocks of independent states. It is also essential to include in the equivalence class measurements that differ infinitesimally. Such extensions are natural in the context of information theory. We shall refer to the above procedure as the “compression of quantum measurement operations”.

The results of [3] are incomplete in several ways and we complete them in the present paper. In particular we give a more precise description of what constitute “equivalent” measurements. We then obtain lower bounds on the amount of classical data I that can be produced by equivalent measurements. Finally we construct measurements that attain the lower bound. Both results apply to general POVMs and in particular to incomplete measurements (for which the POVM elements are not all proportional to one dimensional projectors).

II. PREVIOUS RESULTS

In this section we shall recall the results obtained in [3]. This will serve as a basis for the presentation of our new results in the next section.

Consider a quantum ensemble consisting of states $|\psi_i\rangle$ (in the Hilbert space \mathcal{H} which we assume to be of fi-

nite dimension d throughout the paper), with probabilities p_i ($i = 1, \dots, n$), and a measurement POVM $\mathbf{a} = (a_j)_{j=1, \dots, m}$. We suppose that the measurement maximizes a *fidelity*

$$F(\mathbf{a}) = \sum_i p_i \sum_j \langle \psi_i | a_j | \psi_i \rangle F_{ij},$$

where F_{ij} is the contribution (or *gain*) in the case that on being given state $|\psi_i\rangle$ the POVM hits upon guess j (which happens with probability $\langle \psi_i | a_j | \psi_i \rangle$). Note that this is equivalent to the objective of quantum estimation theory [1,2] to minimize the *cost*. The same minimization problem occurs in the computation of the so-called *quantum rate distortion function*, as defined in [4].

Here the reason for introducing a fidelity is that it allows us to define in an implicit way a class of equivalent measurements. Indeed the F_{ij} encode implicitly a property about which knowledge can be obtained by a measurement. And a measurement that maximizes F is an optimal measurement for this property. One then defines as equivalent all the measurements that maximize F .

It is demonstrated by examples in [3] that the number of outcomes I of the optimal measurement can exceed the von Neumann entropy of the ensemble. But it is proved that if a large number of independent states are available, then one can find an almost optimal measurement that acts collectively on all the copies with logarithm of number of outcomes asymptotically bounded by the von Neumann entropy of the ensemble. This result can be formulated more precisely as follows:

Introduce the density operators $\rho_i = |\psi_i\rangle\langle\psi_i|$, and the average state $\rho = \sum_i p_i \rho_i$. We assume in the sequel that $\rho > 0$ on \mathcal{H} (otherwise pass to the support of ρ) and that \mathcal{H} is finite dimensional. Suppose that a number l of independent states are available. The l states are given by the density operator $\rho_{i^l} = \rho_{i_1} \otimes \dots \otimes \rho_{i_l}$, with probability $p_{i^l} = p_{i_1} \dots p_{i_l}$. Here and in what follows i^l is an abbreviation for a tuple (i_1, \dots, i_l) .

The fidelity for the l independent states is defined as the sum of the individual fidelities

$$F_{i^l, j^l} = \frac{1}{l} \sum_{k=1}^l F_{i_k, j_k}. \quad (1)$$

This is a crucial aspect of the model: the fidelity on blocks is constructed from a sum of fidelities on the individual systems, in fact as the *average* of these fidelities.

We now consider a POVM \mathbf{A} on $\mathcal{H}^{\otimes l}$ and we compute the fidelity for this POVM. This POVM has M outcomes labeled by $\mu = 1, \dots, M$. In order to compute the block fidelity (1) we must associate to each POVM outcome μ a tuple of guesses j_μ^l . Hence the individual POVM elements will be denoted $A_{j_\mu^l}$.

One possible example is the product POVM $\mathbf{a}^{\otimes l}$ which consists of all operators $a_{j^l} = a_{j_1} \otimes \dots \otimes a_{j_l}$. One easily

checks that in this case the fidelity on blocks

$$F(\mathbf{a}^{\otimes l}) = \sum_{i^l} p_{i^l} \sum_{j^l} \langle \psi_{i^l} | a_{j^l} | \psi_{i^l} \rangle F_{i^l j^l}$$

equals the single letter fidelity $F(\mathbf{a})$. In this case the number of outcomes is equal to the maximum possible number of guesses, m^l . However in general the number of guesses M may be smaller than the number of possible tuples. Thus there can be some tuples that are never associated with a POVM element, and hence can never constitute a guess. However even when M is less than the number of possible guesses, we can still compute the average fidelity.

What we are after is a POVM $\mathbf{A} = (A_{j_\mu}^l)_{\mu=1, \dots, M}$ on $\mathcal{H}^{\otimes l}$ whose fidelity $F(\mathbf{A})$ is close to the optimal fidelity F_{opt} and with a minimal number M of outcomes. This will constitute a POVM belonging to the equivalence class for which all the spurious redundancies have been eliminated. The central result of [3] is the construction of such a POVM:

Theorem 1 (Massar, Popescu [3]) *For $\epsilon > 0$ and l large enough there exists a POVM \mathbf{A} with fidelity $F(\mathbf{A}) \geq F_{\text{opt}} - \epsilon$ and*

$$M \leq \exp(l(H(\rho) + \epsilon))$$

many outcomes, where $H(\rho) = -\text{Tr} \rho \log \rho$ is the von Neumann entropy. \square

We can rewrite the fidelity of \mathbf{A} as

$$\begin{aligned} F(\mathbf{A}) &= \sum_{i^l} p_{i^l} \sum_{\mu} \text{Tr}(\rho_{i^l} A_{j_\mu}^l) \frac{1}{l} \sum_{k=1}^l F_{i_k j_\mu k} \\ &= \frac{1}{l} \sum_{k=1}^l \sum_i \sum_j p_i \text{Tr}(\rho_i A_j^{(k)}) F_{ij}, \end{aligned} \quad (2)$$

where (with $[l] = \{1, \dots, l\}$)

$$\begin{aligned} A_j^{(k)} &= \text{Tr}_{\neq k} \left(\left(\rho^{\otimes [l] \setminus k} \otimes \mathbb{1}_k \right) \sum_{\mu: j_{\mu k} = j} A_{j_\mu}^l \right) \\ &= \rho^{-1} \text{Tr}_{\neq k} \left(\rho^{\otimes l} \sum_{\mu: j_{\mu k} = j} A_{j_\mu}^l \right) \\ &= \sqrt{\rho^{-1}} \text{Tr}_{\neq k} \left(\sqrt{\rho}^{\otimes l} \left(\sum_{\mu: j_{\mu k} = j} A_{j_\mu}^l \right) \sqrt{\rho}^{\otimes l} \right) \sqrt{\rho^{-1}}. \end{aligned} \quad (3)$$

To prove the second and third equality recall the defining property of the partial trace on the composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$\forall A \quad \text{Tr} (A \text{Tr}_2 C) = \text{Tr} ((A \otimes \mathbb{1}) C).$$

Note that for all k the $A_j^{(k)}$ ($j = 1, \dots, m$) form a POVM which we shall refer to as *marginals* of \mathbf{A} . The marginals of \mathbf{A} describe the action of the POVM \mathbf{A} restricted to the k 'th state in the block. They will play a central role in what follows.

III. MODEL AND MAIN RESULTS

Theorem 1 is incomplete in several ways: Why do the ensemble states not enter, only their average? Is it important that they are pure? Also, what is the deeper reason that the fidelity matrix does not enter, nor the structure of the optimal measurement? Is the bound on M optimal, or better: under which conditions is it optimal? The results below will help clarify these questions.

We start by analyzing the fidelity constraint on the interesting POVMs: this will lead to a series of conditions (C0–C3) of increasing strength. Theorem 1 lets us start out from the condition

$$|F(\mathbf{A}) - F(\mathbf{a})| \leq \epsilon. \quad (\text{C0})$$

Looking again at (2) we observe that $F(\mathbf{A})$ is an average over the l positions of equally structured quantities: each is an average of the F_{ij} , with probabilities $p_i \text{Tr}(\rho_i A_j^{(k)})$. Thus, assuming that the $|F_{ij}|$ are (without loss of generality) bounded by 1, a POVM \mathbf{A} on $\mathcal{H}^{\otimes l}$ will obtain a fidelity within ϵ of $F(\mathbf{a})$ (for any measurement \mathbf{a} , not only the optimal POVM \mathbf{a} on \mathcal{H}) if, for all k , the distribution

$$\left(\frac{1}{l} \sum_{k=1}^l p_i \text{Tr}(\rho_i A_j^{(k)}) \right)_{ij}$$

is close to $(p_i \text{Tr}(\rho_i a_j))_{ij}$, i.e.

$$\forall k \quad \sum_{ij} \left| \left(\frac{1}{l} \sum_{k=1}^l p_i \text{Tr}(\rho_i A_j^{(k)}) \right) - p_i \text{Tr}(\rho_i a_j) \right| \leq \epsilon. \quad (\text{C1})$$

This will be satisfied if for each position k and each i the corresponding sub-terms are close:

$$\forall k \forall i \quad \sum_j |\text{Tr}(\rho_i A_j^{(k)}) - \text{Tr}(\rho_i a_j)| \leq \epsilon. \quad (\text{C2})$$

And this in turn is satisfied if

$$\forall k \quad \sum_j \|A_j^{(k)} - a_j\| \leq \epsilon. \quad (\text{C3})$$

Here the operator sup norm is used. Proof is by the Hölder inequality for the trace pairing of operators:

$$|\text{Tr}(AB)| \leq \|A\|_1 \cdot \|B\|.$$

Now given any ensemble with average state ρ and a POVM $\mathbf{a} = (a_j)_{j=1, \dots, m}$ a canonical ensemble for ρ can be written down: the states

$$\hat{\rho}_j = \frac{1}{\text{Tr}(\rho a_j)} \sqrt{\rho} a_j \sqrt{\rho},$$

with probabilities $\lambda_j = \text{Tr}(\rho a_j)$.

Note that this ensemble has the property that its “square root” (Holevo [5]) or “pretty good” (Hausladen, Wootters [6]) measurement is exactly \mathbf{a} :

$$a_j = \sqrt{\rho^{-1}} \lambda_j \hat{\rho}_j \sqrt{\rho^{-1}}.$$

Theorem 2 *With the above notation and $\epsilon > 0$, there exists a POVM $\mathbf{A} = (A_{j\mu}^i)_{\mu=1,\dots,M}$ with*

$$M \leq \exp \left(l \left(H(\rho) - \sum_j \lambda_j H(\hat{\rho}_j) \right) + C\sqrt{l} \right)$$

(where C is a constant depending only on ϵ , d and m), and such that

$$\forall k \sum_j \|A_j^{(k)} - a_j\| \leq \epsilon.$$

The characteristic constant in the exponent,

$$I(\lambda; \hat{\rho}) = H(\rho) - \sum_j \lambda_j H(\hat{\rho}_j),$$

is called *entropy defect* of the ensemble (Lebedev and Levitin [7]), or the *quantum mutual information* between a sender producing letter j with probability λ_j and a receiver getting the letter state $\hat{\rho}_j$ (see [8]). It is the difference between the entropy $H(\rho)$ of the ensemble and its *conditional entropy* $H(\hat{\rho}|\lambda) = \sum_j \lambda_j H(\hat{\rho}_j)$.

The theorem is in an asymptotic sense best possible:

Theorem 3 *Let $0 < \epsilon \leq (\lambda_0/2)^2$, with $\lambda_0 = \min_j \lambda_j$. Then for any POVM $\mathbf{A} = (A_{j\mu}^i)_{\mu=1,\dots,M}$ such that*

$$\forall k \sum_j \|A_j^{(k)} - a_j\| \leq \epsilon,$$

one has

$$M \geq \exp \left(l \left(H(\rho) - \sum_j \lambda_j H(\hat{\rho}_j) + \frac{3\epsilon}{\lambda_0^2} \log \frac{2\epsilon}{\lambda_0^2 d} \right) \right).$$

These theorems are proven in the following two sections. They provide answers to the questions at the beginning of this section. By demanding a bit more, namely condition C3 instead of the weaker C0 we find the *optimal* rate of compression for *any* POVM. This improves the previous result (theorem 1) in all cases where the a_j are not all of rank 1. This optimal compression is independent of fidelities, as well as independent of the ensemble structure, except for the average state ρ .

These theorems also answer a question from [3], whether the result of that paper still holds for fidelity measures which cannot be reduced to the form described in the introduction (i.e. an average over certain fixed

numbers, with probabilities $p_i \langle \psi_i | a_j | \psi_i \rangle$), e.g. ones which depend in some nonlinear way on the POVM used. Theorem 2 gives an affirmative answer for all fidelity measures which depend *continuously* on the POVM (to be precise, on its marginals: the definition of the fidelity on blocks as average of the single block fidelities seems to remain essential): an example for this will be discussed in section VII B below.

IV. LOWER BOUND

The proof of theorem 3 rests on some standard facts about von Neumann entropy:

Lemma 4 *Let σ_j be quantum states on \mathcal{H} , λ_j probabilities, and $\sigma = \sum_j \lambda_j \sigma_j$. Then*

$$H(\sigma) \leq H(\lambda) + \sum_j \lambda_j H(\sigma_j).$$

Proof. See [9]: this is just the monotonicity of the mutual information (data processing inequality) under the completely positive and trace preserving map $j \mapsto \sigma_j$ from the commutative algebra generated by the j as mutually orthogonal idempotents to the algebra of linear operators on \mathcal{H} . \square

Lemma 5 *Let $\sigma_1, \dots, \sigma_r$ be states on $\mathcal{H}_1 \otimes \mathcal{H}_2$, with probabilities s_1, \dots, s_r , such that $\sum_i s_i \sigma_i$ is a product state. Then*

$$I(s; \sigma) \geq I(s; \text{Tr}_2 \sigma) + I(s; \text{Tr}_1 \sigma).$$

Proof. This is essentially the sub-additivity of entropy (see [10], p. 23). \square

Lemma 6 *Let $\sigma_1, \dots, \sigma_r$ be states on \mathcal{H} , with probabilities s_1, \dots, s_r , and (J_1, \dots, J_t) a partition of $\{1, \dots, r\}$. Then, denoting*

$$\tilde{s}_j = \sum_{i \in J_j} s_i \text{ and } \tilde{\sigma}_j = \frac{1}{\tilde{s}_j} \sum_{i \in J_j} s_i \sigma_i,$$

it follows that

$$I(s; \sigma) \geq I(\tilde{s}; \tilde{\sigma}).$$

Proof. See [9]: it is another special case of monotonicity, known as *coarse graining*. For a direct proof observe that

$$\sum_j \tilde{s}_j \tilde{\sigma}_j = \sum_i s_i \sigma_i,$$

and by the concavity of von Neumann entropy

$$H(\tilde{\sigma}_j) = H \left(\sum_{i \in I_j} \frac{s_i}{\tilde{s}_j} \sigma_i \right) \geq \sum_{i \in I_j} \frac{s_i}{\tilde{s}_j} H(\sigma_i).$$

\square

Lemma 7 Let ρ, σ be states on \mathcal{H} , $d = \dim \mathcal{H}$, and $\|\rho - \sigma\|_1 \leq \alpha \leq 1/2$. Then

$$|H(\rho) - H(\sigma)| \leq -\alpha \log \frac{\alpha}{d}.$$

Proof. See [10], p. 22. \square

Now we are ready for

Proof of theorem 3. On $\mathcal{H}^{\otimes l}$ consider any POVM $\mathbf{A} = (A_{j_\mu^l})_{\mu=1, \dots, M}$ which satisfies the hypothesis of the theorem. Then, denoting $\Lambda_\mu = \text{Tr}(\rho^{\otimes l} A_{j_\mu^l})$ and

$$\hat{\rho}_\mu^l = \frac{1}{\Lambda_\mu} \sqrt{\rho}^{\otimes l} A_{j_\mu^l} \sqrt{\rho}^{\otimes l},$$

we find

$$\begin{aligned} \log M &\geq H(\Lambda) \geq H(\rho^{\otimes l}) - \sum_\mu \Lambda_\mu H(\hat{\rho}_\mu^l) \\ &= I(\Lambda; \hat{\rho}^l) \geq \sum_{k=1}^l I(\Lambda^{(k)}; \hat{\rho}^{(k)}), \end{aligned}$$

using lemmas 4, 5, and 6, with the marginal distributions given by

$$\Lambda_j^{(k)} = \text{Tr}(\rho A_j^{(k)})$$

and the marginal channel states

$$\hat{\rho}_j^{(k)} = \frac{1}{\Lambda_j^{(k)}} \sqrt{\rho} A_j^{(k)} \sqrt{\rho}.$$

By the hypothesis we have

$$\|\Lambda^{(k)} - \lambda\|_1 \leq \epsilon,$$

and consequently for every k and j

$$\|\hat{\rho}_j^{(k)} - \hat{\rho}_j\|_1 \leq \frac{2}{\lambda_0^2} \epsilon.$$

Thus we can estimate for every k :

$$\begin{aligned} I(\Lambda^{(k)}; \hat{\rho}^{(k)}) &= H(\rho) - \sum_j \Lambda_j^{(k)} H(\hat{\rho}_j^{(k)}) \\ &\geq H(\rho) - \sum_j \lambda_j H(\hat{\rho}_j) \\ &\quad - \epsilon \log d + \frac{2\epsilon}{\lambda_0^2} \log \frac{2\epsilon}{\lambda_0^2 d}, \end{aligned}$$

where we have used lemma 7, and we are done. \square

V. THRIFTY MEASUREMENTS

We will prove theorem 2 in several steps (propositions 9, 10, 11, and 12 below). The strategy is as follows: we construct a series of sub-POVMs² \mathbf{B} , \mathbf{C} , \mathbf{D} ,

²A *sub-POVM* is a POVM except for the weaker condition that the sum of its elements is only *upper bounded* by $\mathbb{1}$.

and \mathbf{E} , each in turn satisfying the condition C3 (which we demonstrate for didactical reasons even though this is not necessary for the ultimate proof), and of increasing regularity. The last step to construct $\mathbf{A} = (A_{j_\mu^l})_{\mu=1, \dots, M}$ is a random selection argument with a novel large deviation probability estimate.

To do this we have first to review the concepts of typical subspace and conditional typical subspace, in the form of [11]:

For a state ρ fix eigenstates e_1, \dots, e_d and define for $\delta > 0$ the *typical projector* as

$$\Pi_{\rho, \delta}^l = \sum_{t^l \text{ with } |\sum_{k=1}^l e_{t_k} - l\rho| \leq \delta \sqrt{l} \sqrt{\rho(\mathbb{1} - \rho)}} e_{t_1} \otimes \dots \otimes e_{t_l}.$$

For a collection of states $\hat{\rho}_j$, $j = 1, \dots, m$, and $j^l \in [m]^l$ define the *conditional typical projector* as

$$\Pi_{\hat{\rho}, \delta}^l(j^l) = \bigotimes_j \Pi_{\hat{\rho}_j, \delta}^{I_j},$$

where $I_j = \{k : j_k = j\}$ and $\Pi_{\hat{\rho}_j, \delta}^{I_j}$ is meant to denote the typical projector of the state $\hat{\rho}_j$ in the positions given by I_j in the tensor product of l factors. From [11] we cite the following properties of these projectors:

$$\text{Tr} \Pi_{\rho, \delta}^l \leq \exp \left(lH(\rho) + Kd\delta\sqrt{l} \right), \quad (4)$$

$$\text{Tr} \Pi_{\rho, \delta}^l \geq \left(1 - \frac{d}{\delta^2} \right) \exp \left(lH(\rho) - Kd\delta\sqrt{l} \right), \quad (5)$$

$$\text{Tr} \Pi_{\hat{\rho}, \delta}^l(j^l) \leq \exp \left(lH(\hat{\rho}|P_{j^l}) + Kmd\delta\sqrt{l} \right), \quad (6)$$

$$\text{Tr} \Pi_{\hat{\rho}, \delta}^l(j^l) \geq \left(1 - \frac{md}{\delta^2} \right) \exp \left(lH(\hat{\rho}|P_{j^l}) + Kmd\delta\sqrt{l} \right), \quad (7)$$

for an absolute constant $K > 0$, and the empirical distribution P_{j^l} of letters j in the word j^l :

$$P_{j^l}(j) = \frac{N(j|j^l)}{l} = \frac{\# \text{ of occurrences of } j \text{ in } j^l}{l}.$$

Also from [11]:

$$\text{Tr}(\rho^{\otimes l} \Pi_{\rho, \delta}^l) \geq 1 - \frac{d}{\delta^2}, \quad (8)$$

$$\text{Tr}(\hat{\rho}_{j^l} \Pi_{\hat{\rho}, \delta}^l(j^l)) \geq 1 - \frac{md}{\delta^2}, \quad (9)$$

with r denoting the minimal eigenvalue of ρ .

To end this review observe the following important operator estimates:

$$\Pi_{\rho, \delta}^l \geq \Pi_{\rho, \delta'}^{[l] \setminus k} \otimes \mathbb{1}, \quad (10)$$

$$\Pi_{\hat{\rho}, \delta}^l(j^l) \geq \Pi_{\hat{\rho}, \delta'}^{[l] \setminus k}(j^{[l] \setminus k}) \otimes \mathbb{1}, \quad (11)$$

where $\delta' = \delta - 1/r \geq \delta/2$, if we assume $\delta \geq 2/r$. Inequalities (10) and (11) will be used in conjunction with the following lemma:

Lemma 8 Let C be a positive operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$, Π a projector on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and Π_0 a projector on \mathcal{H}_2 such that $\Pi \geq \mathbb{1} \otimes \Pi_0$. Then

$$\text{Tr}_2(\Pi C \Pi) \geq \text{Tr}_2((\mathbb{1} \otimes \Pi_0)C(\mathbb{1} \otimes \Pi_0)).$$

Proof. Because of $\Pi(\mathbb{1} \otimes \Pi_0) = \mathbb{1} \otimes \Pi_0$ we may assume that $C = \Pi C \Pi$. Thus we have to prove that

$$\text{Tr}_2 C \geq \text{Tr}_2((\mathbb{1} \otimes \Pi_0)C(\mathbb{1} \otimes \Pi_0)).$$

But this is equivalent to

$$\forall A \geq 0 \quad \text{Tr}((A \otimes \mathbb{1})C) \geq \text{Tr}(A \otimes \Pi_0 C),$$

which in turn is equivalent to $A \otimes \mathbb{1} \geq A \otimes \Pi_0$, and this is obvious. \square

Define the following operators (with ρ and $\hat{\rho}_j$ as in section III):

$$B_{j^l} = \sqrt{\rho^{-1}}^{\otimes l} \Pi_{\hat{\rho}, \delta}^l(j^l) \sqrt{\rho}^{\otimes l} a_{j^l} \sqrt{\rho}^{\otimes l} \Pi_{\hat{\rho}, \delta}^l(j^l) \sqrt{\rho^{-1}}^{\otimes l}.$$

Intuitively this means to confine the a_{j^l} to the range of the conditional typical projector $\Pi_{\hat{\rho}, \delta}^l(j^l)$.

Proposition 9 1. $0 \leq B_{j^l} \leq a_{j^l}$.

$$2. \text{Tr}(\rho^{\otimes l} B_{j^l}) \geq (1 - \frac{md}{\delta^2}) \text{Tr}(\rho^{\otimes l} a_{j^l}).$$

$$3. \sqrt{\rho} a_j \sqrt{\rho} - \Delta_j^1 \leq \sqrt{\rho} B_j^{(k)} \sqrt{\rho} \leq \sqrt{\rho} a_j \sqrt{\rho}, \text{ with } \Delta_j^1 \geq 0 \text{ and } \text{Tr} \Delta_j^1 \leq \lambda_j \frac{md}{\delta^2}.$$

$$4. \forall k \sum_j \|\sqrt{\rho} B_j^{(k)} \sqrt{\rho} - \sqrt{\rho} a_j \sqrt{\rho}\|_1 \leq \frac{md}{\delta^2}.$$

$$5. \forall k \sum_j \|B_j^{(k)} - a_j\| \leq \frac{md}{r\delta^2}.$$

Proof. 1. is equivalent to $\sqrt{\rho}^{\otimes l} B_{j^l} \sqrt{\rho}^{\otimes l} \leq \sqrt{\rho}^{\otimes l} a_{j^l} \sqrt{\rho}^{\otimes l}$, which is immediate from the definition.

2. is essentially equation 9.

3. follows from 1. and 2.

Finally, 4. and 5. are easy consequences of 3. \square

Defining the operators

$$C_{j^l} = \Pi_{\hat{\rho}, \delta}^l B_{j^l} \Pi_{\hat{\rho}, \delta}^l,$$

i.e. restricting the B_{j^l} to the range of the typical projector $\Pi_{\hat{\rho}, \delta}^l$, we find

Proposition 10 1. $\text{Tr}(\rho^{\otimes l} C_{j^l}) \leq \text{Tr}(\rho^{\otimes l} B_{j^l})$.

$$2. \sqrt{\rho} a_j \sqrt{\rho} - \Delta_j^2 \leq \sqrt{\rho} C_j^{(k)} \sqrt{\rho} \leq \sqrt{\rho} a_j \sqrt{\rho} + \Delta_j^2, \text{ with } \Delta_j^2 \geq 0, \Delta_j^2 = \sum_j \Delta_j^2, \text{ and } \text{Tr} \Delta_j^2 \leq \lambda_j \frac{md+4d}{\delta^2}.$$

$$3. \forall k \sum_j \|\sqrt{\rho} C_j^{(k)} \sqrt{\rho} - \sqrt{\rho} B_j^{(k)} \sqrt{\rho}\|_1 \leq \frac{m^2+4md}{\delta^2}.$$

$$4. \forall k \sum_j \|C_j^{(k)} - B_j^{(k)}\| \leq \frac{m^2 d + 4md}{r\delta^2}.$$

Proof. 1. follows from $\Pi_{\hat{\rho}, \delta}^l \rho^{\otimes l} \Pi_{\hat{\rho}, \delta}^l \leq \rho^{\otimes l}$, and the definition.

To prove 2., we first do the lower bound (the other follows from this straightforwardly):

$$\begin{aligned} \sqrt{\rho} C_j^{(k)} \sqrt{\rho} &= \text{Tr}_{\neq k} \left(\sum_{j^l: j_k=j} \sqrt{\rho}^{\otimes l} C_{j^l} \sqrt{\rho}^{\otimes l} \right) \\ &= \text{Tr}_{\neq k} \left(\Pi_{\rho, \delta}^l \left(\sum_{j^l: j_k=j} \sqrt{\rho}^{\otimes l} B_{j^l} \sqrt{\rho}^{\otimes l} \right) \Pi_{\rho, \delta}^l \right) \\ &\geq \text{Tr}_{\neq k} \left(\left(\sum_{j^l: j_k=j} \sqrt{\rho}^{\otimes l} a_{j^l} \sqrt{\rho}^{\otimes l} - \Delta_{j^l} \right) (\Pi_{\rho, \delta}^{[l] \setminus k} \otimes \mathbb{1}) \right) \\ &= \sqrt{\rho} a_j \sqrt{\rho} \text{Tr}(\rho^{\otimes [l] \setminus k} \Pi_{\rho, \delta}^{[l] \setminus k}) - \Delta \\ &= \sqrt{\rho} a_j \sqrt{\rho} - \Delta_j^2, \end{aligned}$$

where the inequality is with $\Delta_{j^l} \geq 0$, $\text{Tr} \Delta_{j^l} \leq \frac{md}{\delta^2} \lambda_{j^l}$ (by proposition 9.1. and 2.), and by lemma 8. Hence the subsequent equalities are valid with

$$\text{Tr} \Delta \leq \lambda_j \frac{md}{\delta^2}, \text{ with } \Delta = \sum_{j^l: j_k=j} \Delta_{j^l},$$

and

$$\Delta_j^2 = \Delta + \sqrt{\rho} a_j \sqrt{\rho} \left(1 - \text{Tr}(\rho^{\otimes [l] \setminus k} \Pi_{\rho, \delta}^{[l] \setminus k}) \right).$$

By equation 8 we conclude $\text{Tr} \Delta_j^2 \leq \lambda_j \frac{md+4d}{\delta^2}$.

Finally, 3. and 4. are easy consequences of 2. \square

Now with the probabilities $\lambda_j = \text{Tr}(\rho a_j)$ define the set of typical sequences

$$\mathcal{T}_\delta^l = \{j^l : \forall j \quad |N(j|j^l) - l\lambda_j| \leq \delta \sqrt{l} \sqrt{\lambda_j(1-\lambda_j)}\}.$$

The next simplification is to use only operators of our sub-POVM \mathbf{C} with typical j^l : define the sub-POVM \mathbf{D} to consist of the C_{j^l} for $j^l \in \mathcal{T}_\delta^l$, i.e. $\mathbf{D} = (C_{j^l})_{j^l \in \mathcal{T}_\delta^l}$.

Proposition 11 1. $\lambda^l(\mathcal{T}_\delta^l) =: S \geq 1 - \frac{m}{\delta^2}$.

$$2. \text{Tr}(\rho^{\otimes l} D_{j^l}) \geq \left(1 - \frac{2m^3 d}{r^2 \delta^2}\right) \text{Tr}(\rho^{\otimes l} a_{j^l}).$$

$$3. \sqrt{\rho} D_j^{(k)} \sqrt{\rho} = \sqrt{\rho} C_j^{(k)} \sqrt{\rho} - \Delta_j^3, \text{ with } \Delta_j^3 \geq 0, \sum_j \text{Tr} \Delta_j^3 \leq m/\delta^2.$$

$$4. \forall k \sum_j \|\sqrt{\rho} D_j^{(k)} \sqrt{\rho} - \sqrt{\rho} C_j^{(k)} \sqrt{\rho}\|_1 \leq \frac{m}{\delta^2}.$$

$$5. \forall k \sum_j \|D_j^{(k)} - C_j^{(k)}\| \leq \frac{m}{r\delta^2}.$$

Proof. 1. follows from Chebyshev's inequality (compare [11]).

2. is seen as follows: with the eigenstates e_t of ρ define

$$\tilde{\rho}_j = E(\hat{\rho}_j) = \sum_t e_t \hat{\rho}_j e_t,$$

with the conditional expectation E . Then it is obvious that

$$\text{Tr} (\hat{\rho}_{j^l} \Pi_{\rho,\delta}^l) = \text{Tr} (\tilde{\rho}_{j^l} \Pi_{\rho,\delta}^l).$$

From the definitions it can be directly verified that, with $\tilde{\rho} = \frac{1}{l} \sum_k \tilde{\rho}_{jk}$,

$$\Pi_{\rho,\delta}^l \geq \Pi_{\tilde{\rho}, \frac{r}{m} \delta \sqrt{m}},$$

hence by [11], lemma V.9

$$\text{Tr} (\hat{\rho}_{j^l} \Pi_{\rho,\delta}^l) \geq 1 - \frac{m^3 d}{r^2 \delta^2},$$

and with proposition 9.2. the claim follows. For 3. observe

$$\sqrt{\rho} C_j^{(k)} \sqrt{\rho} - \sqrt{\rho} D_j^{(k)} \sqrt{\rho} = \sum_{j^l \notin \mathcal{T}_\delta^l: j_k=j} \text{Tr}_{\neq k} \sqrt{\rho}^{\otimes l} C_{j^l} \sqrt{\rho}^{\otimes l},$$

and denoting the r.h.s by Δ_j^3 , the claim follows from 1., observing that

$$\text{Tr} \sqrt{\rho}^{\otimes l} C_{j^l} \sqrt{\rho}^{\otimes l} \leq \lambda_{j^l},$$

by propositions 9.1. and 10.1.

Again, 4. and 5. are easy consequences. \square

We shall use the probability distribution Λ on \mathcal{T}_δ^l , with

$$\Lambda_{j^l} = \frac{1}{S} \lambda_{j^l}^l.$$

Observe that

$$\omega = \sum_{j^l \in \mathcal{T}_\delta^l} \sqrt{\rho}^{\otimes l} D_{j^l} \sqrt{\rho}^{\otimes l} = \rho^{\otimes l} - \Delta^4,$$

with

$$\text{Tr} \Delta^4 \leq \frac{(m+1)(d+1)}{\delta^2} =: c.$$

Introducing

$$\alpha = \left(1 - \frac{d}{\delta^2}\right) \exp(-lH(\rho) - Kd\delta\sqrt{l})$$

(so that $\Pi_{\rho,\delta}^l \rho^{\otimes l} \Pi_{\rho,\delta}^l \geq \alpha \Pi_{\rho,\delta}^l$), we can construct the subspace spanned by the eigenvectors of M corresponding to eigenvalues at least $c\alpha$. With its projection Π we have $\Pi\omega\Pi \geq c\alpha\Pi$. This implies

$$\text{Tr} (\omega(\Pi_{\rho,\delta}^l - \Pi)) \leq c,$$

hence because of $\text{Tr} \omega \Pi_{\rho,\delta}^l \geq 1 - c$

$$\text{Tr} \omega \Pi \geq 1 - 2c. \quad (12)$$

Now, define the sub-POVM \mathbf{E} by

$$E_{j^l} = \sqrt{\rho^{-1}}^{\otimes l} \Pi \sqrt{\rho}^{\otimes l} D_{j^l} \sqrt{\rho}^{\otimes l} \Pi \sqrt{\rho^{-1}}^{\otimes l},$$

for $j^l \in \mathcal{T}_\delta^l$.

Proposition 12 For $j^l \in \mathcal{T}_\delta^l$:

1. $\text{Tr} (\rho^{\otimes l} E_{j^l}) \leq \text{Tr} (\rho^{\otimes l} D_{j^l})$.
2. $\forall k \sum_j \|\sqrt{\rho} E_j^{(k)} \sqrt{\rho} - \sqrt{\rho} D_j^{(k)} \sqrt{\rho}\|_1 \leq 2mc$.
3. $\forall k \sum_j \|E_j^{(k)} - D_j^{(k)}\|_1 \leq 2mc/r$.

Proof. 1. is obvious, and 3. follows from 2.

To prove 2., first calculate

$$\begin{aligned} \sqrt{\rho} E_j^{(k)} \sqrt{\rho} &= \text{Tr}_{\neq k} \left(\sum_{j^l \in \mathcal{T}_\delta^l: j_k=j} \sqrt{\rho}^{\otimes l} E_{j^l} \sqrt{\rho}^{\otimes l} \right) \\ &= \text{Tr}_{\neq k} \left(\sum_{j^l \in \mathcal{T}_\delta^l: j_k=j} \Pi \sqrt{\rho}^{\otimes l} D_{j^l} \sqrt{\rho}^{\otimes l} \Pi \right) \\ &=: \lambda_j \text{Tr}_{\neq k} \Pi \omega_{kj} \Pi, \end{aligned}$$

with

$$\omega_{kj} = \frac{1}{\lambda_j} \sum_{j^l \in \mathcal{T}_\delta^l: j_k=j} \sqrt{\rho}^{\otimes l} D_{j^l} \sqrt{\rho}^{\otimes l}.$$

Observe that by equation 12

$$\text{Tr} \omega_{kj} \Pi \geq 1 - \frac{2c}{\lambda_j}.$$

But because of

$$\begin{aligned} \Pi_{\rho,\delta}^l \omega_{kl} \Pi_{\rho,\delta}^l - \Pi \omega_{kl} \Pi &= (\Pi_{\rho,\delta}^l - \Pi) \omega_{kl} \Pi_{\rho,\delta}^l \\ &\quad + \Pi \omega_{kl} (\Pi_{\rho,\delta}^l - \Pi), \end{aligned}$$

we get

$$\begin{aligned} \|\Pi_{\rho,\delta}^l \omega_{kl} \Pi_{\rho,\delta}^l - \Pi \omega_{kl} \Pi\|_1 &\leq \|(\Pi_{\rho,\delta}^l - \Pi) \omega_{kl} \Pi_{\rho,\delta}^l\|_1 \\ &\quad + \|\Pi \omega_{kl} (\Pi_{\rho,\delta}^l - \Pi)\|_1 \\ &\leq 2 \text{Tr} (\omega_{kl} (\Pi_{\rho,\delta}^l - \Pi)) \\ &\leq 2c/\lambda_j, \end{aligned}$$

thus we conclude

$$\begin{aligned} \sqrt{\rho} E_j^{(k)} \sqrt{\rho} &= \text{Tr}_{\neq k} \Pi_{\rho,\delta}^l \lambda_j \omega_{kj} \Pi_{\rho,\delta}^l + \Delta_j^5 \\ &= \sqrt{\rho} D_j^{(k)} \sqrt{\rho} + \Delta_j^5, \end{aligned}$$

where $\|\Delta_j^5\|_1 \leq 2c$. \square

The proof of the theorem will now be completed by a random selection of a sufficient number of elements from \mathbf{E} . We invoke a result from [12]:

Lemma 13 Let X_1, \dots, X_M be i.i.d. random variables with values in the algebra $\mathcal{L}(\mathcal{K})$ of linear operators on \mathcal{K} , which are bounded between 0 and $\mathbf{1}$. Assume that the average $\mathbb{E}X_\mu = \sigma \geq s\mathbf{1}$. Then for every $\eta > 0$

$$\Pr \left\{ \frac{1}{M} \sum_{\mu=1}^M X_\mu \not\geq (1+\eta)\sigma \right\} \leq \dim \mathcal{K} \exp \left(-M \frac{\eta^2 s}{2 \ln 2} \right).$$

With this we can now finish

Proof of theorem 2. Starting from the POVM $\mathbf{a}^{\otimes l}$ construct the sub-POVMs \mathbf{B} , \mathbf{C} , \mathbf{D} , and \mathbf{E} , as above.

Define i.i.d. random variables J_1, \dots, J_M with values in \mathcal{T}_δ^l such that

$$\Pr\{J_\mu = j^l\} = \Lambda_{j^l}, \quad \mu = 1, \dots, M.$$

These define operator valued random variables

$$X_\mu = \frac{S}{\lambda_{J_\mu}} \sqrt{\rho}^{\otimes l} E_{J_\mu} \sqrt{\rho}^{\otimes l}, \quad \mu = 1, \dots, M.$$

Observe that for all μ

$$\mathbb{E}X_\mu = \Pi\omega\Pi \leq \omega \leq \Pi_{\rho,\delta}^l \rho^{\otimes l} \Pi_{\rho,\delta}^l \leq \rho^{\otimes l},$$

and that $X_\mu \geq 0$ with $\text{Tr } X_\mu \leq 1$. Furthermore, since

$$\sqrt{\rho}^{\otimes l} E_{j^l} \sqrt{\rho}^{\otimes l} = \lambda_{j^l} \Pi \Pi_{\hat{\rho},\delta}^l(j^l) \hat{\rho}_{j^l} \Pi_{\hat{\rho},\delta}^l(j^l) \Pi,$$

we have $X_\mu \leq \beta \Pi$ with

$$\beta = \exp\left(-lH(\hat{\rho}|\lambda) + Kmd\delta\sqrt{l}\right).$$

Most importantly we find

$$\mathbb{E}X_\mu = \Pi\omega\Pi \geq c\alpha\Pi.$$

Apply lemma 13 to the variables $\beta^{-1}X_\mu$ to find

$$\begin{aligned} \Pr\left\{\frac{1}{M} \sum_{\mu=1}^M X_\mu \not\leq (1+\eta)\Pi\omega\Pi\right\} \\ \leq \text{Tr } \Pi \exp\left(-M \frac{\eta^2 c \alpha}{2\beta \ln 2}\right). \end{aligned} \quad (13)$$

Define $Y_\mu^{[jk]} = \text{Tr}_{\neq k} X_\mu$ if $J_{\mu k} = j$ and 0 otherwise. Observe that

$$\mathbb{E}Y_\mu^{[jk]} = \sqrt{\rho} E_j^{(k)} \sqrt{\rho},$$

so by propositions 9.2., 10.2., 11.2., and 12.2.

$$\|\mathbb{E}Y_\mu^{[jk]} - \sqrt{\rho} a_j \sqrt{\rho}\|_1 \leq 2c + \frac{m}{\delta^2} + \frac{m^2 d + 4md}{\delta^2} + \frac{md}{\delta^2} =: \tilde{c}.$$

Thus, by the operator Chebyshev inequality [12]

$$\Pr\left\{\left\|\sum_{\mu} Y_\mu^{[jk]} - M\sqrt{\rho} a_j \sqrt{\rho}\right\|_1 > M\tilde{c} + \delta\sqrt{M}\sqrt{l}\right\} \leq \frac{d}{l\delta^2}. \quad (14)$$

In case that the sum of the right hand sides of the probability estimates from equations (13) and (14) ($j = 1, \dots, m$, $k = 1, \dots, l$) is less than 1 — which can be forced by choosing

$$\delta > \sqrt{2md} \text{ and } M > \frac{2 \ln 2(1 - \log \alpha) \beta}{\eta^2 c} \frac{\beta}{\alpha} \quad (15)$$

— there are actual values $J_1 = j_1^l, \dots, J_M = j_M^l$ such that

$$\frac{1}{M} \sum_{\mu} \frac{1}{\Lambda_{j_\mu^l}} \sqrt{\rho}^{\otimes l} E_{j_\mu^l} \sqrt{\rho}^{\otimes l} \leq (1+\eta) \Pi_{\rho,\delta}^l \rho^{\otimes l} \Pi_{\rho,\delta}^l, \quad (16)$$

and

$$\begin{aligned} \forall k \forall j \left\| \frac{1}{M} \sum_{\mu: j_{\mu k} = j} \text{Tr}_{\neq k} \left(\frac{1}{\Lambda_{j_\mu^l}} \sqrt{\rho}^{\otimes l} E_{j_\mu^l} \sqrt{\rho}^{\otimes l} \right) - \sqrt{\rho} a_j \sqrt{\rho} \right\|_1 \\ \leq \tilde{c} + \frac{\delta\sqrt{l}}{\sqrt{M}}. \end{aligned} \quad (17)$$

In this case we may form the following sub-POVM:

$$\begin{aligned} \tilde{A}_{j_\mu^l} &= \frac{1}{(1+\eta)M} \sqrt{\rho^{-1}}^{\otimes l} \left(\frac{S}{\lambda_{j_\mu^l}} \sqrt{\rho}^{\otimes l} E_{j_\mu^l} \sqrt{\rho}^{\otimes l} \right) \sqrt{\rho^{-1}}^{\otimes l} \\ &= \frac{1}{(1+\eta)M} \frac{S}{\lambda_{j_\mu^l}} E_{j_\mu^l}. \end{aligned}$$

First observe that this is indeed a sub-POVM, as by equation (16)

$$\sum_{\mu} \tilde{A}_{j_\mu^l} \leq \Pi_{\rho,\delta}^l.$$

We claim that it satisfies condition C3, more precisely, by equation (17) we find

$$\begin{aligned} \forall k \forall j \left\| \sum_{\mu: j_{\mu k} = j} \text{Tr}_{\neq k} \left(\sqrt{\rho}^{\otimes l} \tilde{A}_{j_\mu^l} \sqrt{\rho}^{\otimes l} \right) - \sqrt{\rho} a_j \sqrt{\rho} \right\|_1 \\ \leq \eta + \tilde{c} + \frac{\delta\sqrt{l}}{\sqrt{M}}. \end{aligned}$$

Distributing the remainder $R = \mathbb{1} - \sum_{\mu} \tilde{A}_{j_\mu^l}$ equally over the operators will give us our desired POVM \mathbf{A} :

$$A_{j_\mu^l} = \tilde{A}_{j_\mu^l} + \frac{1}{M} R.$$

Namely, it is immediate that now (inserting equation 3)

$$\forall k \forall j \left\| \sqrt{\rho} A_j^{(k)} \sqrt{\rho} - \sqrt{\rho} a_j \sqrt{\rho} \right\|_1 \leq (m+1) \left(\eta + \tilde{c} + \frac{\delta\sqrt{l}}{\sqrt{M}} \right),$$

and we are done, by choosing $\eta = \delta^{-2}$, with δ suitably large, and M according to equation (15). \square

VI. EXTENSIONS

Not to encumber the proofs with too many estimates which actually would not contribute to the understanding of the results, we chose to present theorems 2 and 3 in their above form.

However, let us see here how far we can actually get with our theorem 2: we might for example go beyond C3 by requiring

$$\sum_k \sum_j \|A_j^{(k)} - a_j\| \leq \epsilon. \quad (\text{C4})$$

We might go even further, and demand that \mathbf{A} approximates $\mathbf{a}^{\otimes l}$ not only on single factors but also on all subsets of factors, $K \subset \{1, \dots, l\}$, of moderate growing size, say $|K| \leq \nu_l = o(l)$:

$$\sum_{K \subset [l], |K| \leq \nu_l} \sum_{j^K} \|A_{j^K}^{(K)} - a_{j^K}\| \leq \epsilon. \quad (\text{C5})$$

Here

$$A_{j^K}^{(K)} = \text{Tr}_{[l] \setminus K} \left(\left(\rho^{\otimes [l] \setminus K} \otimes \mathbb{1}^{\otimes K} \right) \sum_{\mu: \forall k \in K, j_{\mu k} = j_k} A_{j_\mu}^l \right)$$

is the restriction of \mathbf{A} to the tensor factors K and

$$a_{j^K} = \bigotimes_{k \in K} a_{j_k}$$

is an element of the K -factor POVM $\mathbf{a}^{\otimes K}$.

It turns out that, using slightly stronger estimates for the typical subspaces and typical sequences than those used in section V one can prove

Theorem 14 *With the above notation, there exists a POVM $\mathbf{A} = (A_{j_\mu}^l)_{\mu=1, \dots, M}$ with*

$$M \leq \exp \left(l \left(H(\rho) - \sum_j \lambda_j H(\hat{\rho}_j) \right) + o(l) \right)$$

and satisfying condition C5.

Proof. From [13], lemma I.9, we use the following estimates: for fixed $\hat{\rho}_j$ and ρ there exists a constant $\gamma > 0$ such that

$$\text{Tr}(\rho^{\otimes l} \Pi_{\rho, \delta}^l) \geq 1 - d \cdot e^{-\gamma \delta^2}, \quad (\text{18})$$

$$\text{Tr}(\hat{\rho}_j^l \Pi_{\hat{\rho}_j, \delta}^l(j^l)) \geq 1 - md \cdot e^{-\gamma \delta^2}. \quad (\text{19})$$

Instead of equations (8) and (9), use equations (18) and (19) in all steps of the proof in section V. Then, choosing $\delta = \delta_l$ such that

$$\nu_l = o(\delta_l^2), \quad \delta_l = o(\sqrt{l}),$$

and with $\eta = \exp(-\delta^2)$, the theorem follows. \square

As an immediate corollary we get an improvement of theorem 1:

Theorem 15 *For $\epsilon > 0$ and l large enough there exists a POVM \mathbf{A} satisfying*

$$l \cdot F(\mathbf{A}) \geq l \cdot F_{\text{opt}} - \epsilon$$

and with

$$M \leq \exp(l(H(\rho) + \epsilon))$$

many outcomes. \square

On the other hand, inspection of the proof of theorem 3 shows that it remains valid (up to another $O(\epsilon l)$ in the exponent) under the slightly weaker condition

$$\frac{1}{l} \sum_{k=1}^l \sum_j \|A_j^{(k)} - a_j\| \leq \epsilon. \quad (\text{C2}_{\frac{1}{2}})$$

VII. DISCUSSION

In this article we have shown how to compress quantum measurements. More precisely we have shown how to devise a measurement \mathbf{A} that is close to a certain given POVM \mathbf{a} but that produces a minimum amount of data. This minimum amount of data is equal to $I(\lambda; \hat{\rho}) = H(\rho) - \sum_j \lambda_j H(\hat{\rho}_j)$, where

$$\hat{\rho}_j = \frac{1}{\lambda_j} \sqrt{\rho} a_j \sqrt{\rho}, \quad \lambda_j = \text{Tr} \rho a_j.$$

This result provides a precise measure of how much knowledge about the unknown states is provided by the measurement \mathbf{a} . Namely the amount of knowledge provided by the measurement is equal to the minimal amount of classical data produced measurements \mathbf{A} that are close to \mathbf{a} . This is because the measurement \mathbf{A} resembles the measurement \mathbf{a} , hence provides as much knowledge about the states as \mathbf{a} . But the spurious randomness in data produced by the measurement \mathbf{a} has been removed. Thus we deduce that the amount of meaningful data produced by the measurement \mathbf{a} is $I(\lambda; \hat{\rho})$.

We now consider several questions and lines of inquiry which are suggested by the present results.

A. Information missed by incomplete measurements

Consider a POVM that is not maximally refined. By this we mean that the POVM elements a_j are not all proportional to one dimensional projectors. Such a POVM does not provide maximum knowledge about a quantum state. However at a later stage one can refine the POVM so as to obtain additional knowledge about the state. We would like to know whether carrying out such a sequence of measurements provides the maximum knowledge about the state, or whether there is an irreversible loss of knowledge in such a two step measurement. We

shall argue, using the results presented in this paper, that if the first measurement is carried out in such a way to minimize disturbance to the state, then no knowledge is lost by such a two step procedure.

When the POVM is not maximally refined the amount of meaningful data produced by the measurement is less than if the measurement is maximally refined since the term which one subtracts $\sum_j \lambda_j H(\hat{\rho}_j)$ in $I(\lambda; \hat{\rho})$ vanishes in one case and not in the other. Suppose that after the first measurement one carries out a second measurement \mathbf{b} which is maximally refined. Let us note that after the first measurement, if the state was ρ_i and the outcome was j , one obtains the state σ_{ji} given by the completely positive map

$$\sigma_{ji} = \frac{\sum_\nu V_{j\nu} \rho_i V_{j\nu}^\dagger}{\text{Tr } \rho_i a_j} \quad (20)$$

where $\sum_\nu V_{j\nu}^\dagger V_{j\nu} = a_j$. And the average state if the outcome was j is

$$\sigma_j = \frac{\sum_\nu V_{j\nu} \rho V_{j\nu}^\dagger}{\text{Tr } \rho a_j} \quad (21)$$

Since the second measurement is maximally refined, the amount of meaningful data it produces is equal to $H(\sigma_j)$.

We first consider the case when the completely positive map has only one term in its Kraus representation. In this case $\sigma_j = U_j \sqrt{a_j} \rho \sqrt{a_j} U_j^\dagger$ (where U_j is a unitary matrix). We now show that the amount of meaningful data produced by the second measurement $H(\sigma_j)$ is equal to the amount of data missing from the first measurement $H(\hat{\rho}_j)$. This follows from the fact that σ_j and $\hat{\rho}_j$ have the same spectrum, as they are conjugates. To show this it is sufficient to show that $\sqrt{\rho} a_j \sqrt{\rho}$ and $\sqrt{a_j} \rho \sqrt{a_j}$ are conjugates. Using the notation $B = \sqrt{a_j} \sqrt{\rho}$, we have

$$\begin{aligned} B^\dagger B &= \sqrt{\rho} a_j \sqrt{\rho}, \\ BB^\dagger &= \sqrt{a_j} \rho \sqrt{a_j}. \end{aligned}$$

Introducing the polar decomposition $B = U|B|$ (where U is unitary and $|B| = \sqrt{B^\dagger B}$), we find

$$BB^\dagger = U|B|^2 U^\dagger = U(B^\dagger B)U^\dagger$$

which is what we needed to show.

Thus in the case where the Kraus representation of the completely positive map (20) contains only one term, the deficit in the amount of meaningful data produced by the first measurement is exactly equal to the amount of meaningful data obtained by the second (maximally refined) measurement. It thus appears that making an incomplete measurement which is such that the Kraus representation of the measurement operation contains only one term for each POVM element does not give rise to an irreversible loss of knowledge. Rather the knowledge is still present and can be accessed by a second more refined measurement.

The case when the Kraus representation of the completely positive map (20) contains only one term corresponds to the situation in which one disturbs as little as possible the quantum state. On the other hand when the Kraus representation contains more than one term, one easily checks on examples that the amount of information obtained by the second measurement bears no relation to the amount of information obtained by the first measurement. This is because the map can either add noise to the state or take away information.

The above discussion raises an interesting question concerning the amount of information transferred to a state or taken away from the state by a completely positive map. The approach developed in this paper may illuminate this question and we hope to report on this in a future paper.

B. Relation to Holevo's bound

Consider an ensemble of states $\{\sigma_i, \mu_i\}$ whose average is $\sum_i \mu_i \sigma_i = \rho$ and consider a POVM with elements a_j . We define random variables X, Y with joint distribution

$$\Pr\{X = i, Y = j\} = \mu_i \text{Tr}(\sigma_i a_j). \quad (22)$$

They describe the joint probability that state σ_i occurred in the ensemble and measurement outcome j occurred.

Holevo's bound [14] states that the mutual entropy between a source with ensemble $\{\sigma_i, \mu_i\}$ and a measurement a_j is bounded by the entropy defect of the ensemble:

$$I(X \wedge Y) \leq I(\mu; \sigma) = H(\rho) - \sum_i \mu_i H(\sigma_i). \quad (23)$$

Note that Holevo's bound is a function only of the ensemble $\{\sigma_i, \mu_i\}$, and the measurement plays no role in the bound. On the other hand in the present paper the ensemble plays a secondary role, and we have considered how the joint distribution of X, Y changes when one changes the measurement. In order to make connection with Holevo's bound we shall use a trick that allows us to switch the role of ensemble and measurement.

Let us denote the triple consisting of the states σ_i , the probabilities μ_i and the POVM elements a_j by

$$M_{\sigma, \mu, \mathbf{a}} = \{\sigma_i, \mu_i, a_j\}.$$

We now construct a second triple

$$N_{\hat{\rho}, \lambda, \mathbf{S}} = \{\hat{\rho}_j, \lambda_j, S_i\}$$

canonically associated with the first. In this second triple the states are

$$\hat{\rho}_j = \frac{1}{\lambda_j} \sqrt{\rho} a_j \sqrt{\rho}$$

and their probabilities are $\lambda_j = \text{Tr } \rho a_j$. The POVM elements S_i of the second triple are the "pretty good" measurement of the ensemble $\{\sigma_i, \mu_i\}$:

$$S_i = \sqrt{\rho^{-1}} \mu_i \sigma_i \sqrt{\rho^{-1}}.$$

We call these two triples canonically associated for two reasons. First the average of the states is the same $\sum_i \mu_i \sigma_i = \sum_j \lambda_j \hat{\rho}_j = \rho$. Second the probability that state σ_i occurred and measurement outcome j occurred the first triple is equal to the probability that state $\hat{\rho}_j$ occurred and the measurement outcome i occurred in the second triple:

$$\Pr\{X = i, Y = j\} = \mu_i \text{Tr}(\sigma_i a_j) = \lambda_j \text{Tr}(S_i \hat{\rho}_j). \quad (24)$$

Using the relation between these two triples and in particular eq. (24) we can write two forms of Holevo's bound. The first is equation (23), the second is

$$I(X \wedge Y) \leq I(\hat{\rho}; \lambda) = H(\rho) - \sum_j \lambda_j H(\hat{\rho}_j). \quad (25)$$

Thus for a given triple, say $M_{\sigma, \mu, \mathbf{a}}$, we can derive two bounds on the mutual information, the first (23) depends only the ensemble $\{\sigma_i, \mu_i\}$, the second (25) depends only on the average state ρ and the POVM \mathbf{a} .

In order to establish a connections between the present work and Holevo's bound, we use the second form of Holevo's bound, equation (25). Let us first note that theorem 2 shows that one can always devise a measurement \mathbf{A} acting collectively on many independent states whose marginals are close to the POVM \mathbf{a} and with a number of outcomes equal to the right hand side of (24). Thus Holevo's bound and theorem 2 are consistent.

However we can go further and use (24) together with theorem 2 to derive Holevo's bound. Let $\{\hat{\rho}_j, \lambda_j\}$ be any ensemble of states with average $\rho = \sum_j \lambda_j \hat{\rho}_j$, and (S_i) a POVM. With random variables X, Y as defined in the second equality in (22), Holevo's bound is equivalent to (25). Let us denote the classical mutual information $I(X \wedge Y)$ by $I(\{\hat{\rho}_j, \lambda_j\} \wedge (S_i))$. Now we revert the argument from the beginning of this subsection and invent the POVM \mathbf{a} and the ensemble $\{\sigma_i, \mu_i\}$, so that the first equality in (24) is satisfied. In particular we get

$$I(\{\sigma_i, \mu_i\} \wedge (a_j)) = I(\{\hat{\rho}_j, \lambda_j\} \wedge (S_i)),$$

and what we have to prove transforms into

$$I(\{\sigma_i, \mu_i\} \wedge (a_j)) \leq I(\lambda; \hat{\rho}).$$

Here our theorem 2 comes in: define, for any POVM \mathbf{a} , the fidelity function

$$F(\mathbf{a}) = I(\{\sigma_i, \mu_i\} \wedge (a_j)),$$

and for the POVM \mathbf{A} on $\mathcal{H}^{\otimes l}$ the fidelity on blocks

$$F(\mathbf{A}) = \frac{1}{l} \sum_{k=1}^l I(\{\sigma_i, \mu_i\} \wedge (A_j^{(k)})).$$

Observe that this is a nonlinear continuous function of the POVM (the ensemble $\{\sigma_i, \mu_i\}$ we now consider as fixed).

By theorem 2 we find, for $\epsilon > 0$ and large enough l :

$$\begin{aligned} I(\lambda; \hat{\rho}) + \epsilon &\geq \frac{1}{l} \log M \\ &\geq \frac{1}{l} I(\{\sigma_i, \mu_i\}^{\otimes l} \wedge A_{j_\mu}^l) \\ &= \frac{1}{l} I(X^l \wedge Y) \\ &\quad \left[\Pr\{X^l = i^l, Y = \mu\} = \mu_{i^l} \text{Tr}(\sigma_{i^l} A_{j_\mu}^l) \right] \\ &\geq \frac{1}{l} \sum_{k=1}^l I(X_k \wedge Y) \\ &\geq \frac{1}{l} \sum_{k=1}^l I(X_k \wedge f_k(Y)) \quad \left[f_k(\mu) = j_{\mu k} \right] \\ &= \frac{1}{l} \sum_{k=1}^l I(\{\sigma_i, \mu_i\} \wedge (A_j^{(k)})) \\ &= F(\mathbf{A}) \\ &\geq F(\mathbf{a}) - \epsilon \\ &= I(\{\sigma_i, \mu_i\} \wedge (a_j)) - \epsilon. \end{aligned}$$

(Only classical information inequalities have been used: the second line is by data processing, the fourth from independence of the X_k , the fifth by data processing again). Because $\epsilon > 0$ was arbitrary, we are done.

C. Data vs. information

The above discussion concerning the Holevo bound can be used to address the relation between *data* and (*mutual*) *information*.

Holevo's bound as usually presented is a function only of the ensemble $\{\sigma_i, \mu_i\}$ of states emitted by a source. Maximizing over the measurement, with fixed ensemble, yields the accessible information at fixed ensemble $I_{\text{acc}}(\mu; \sigma)$. It was shown in [14] that the accessible information attains the Holevo bound if and only if all the states that compose the ensemble commute. Furthermore this difference remains even asymptotically when one considers measurements on many independent states emitted by the source because (see [15])

$$I_{\text{acc}}(\mu^{\otimes l}; \sigma^{\otimes l}) = l \cdot I_{\text{acc}}(\mu; \sigma).$$

On the other hand it is known that one can carry out block coding and construct an ensemble whose marginals are such that they are distributed in the same way as the original ensemble, such that for this ensemble the accessible information approaches with the Holevo bound: see [8,16,11].

Let us now transcribe these results in terms of measurements, using the second form of Holevo's bound discussed above. If one keeps the measurement \mathbf{a} fixed and maximizes over the ensemble (with the average state ρ fixed),

one reaches the accessible information at fixed measurement and fixed average state, which we denote $J_\rho(\mathbf{a})$. It follows from the above discussion that $J_\rho(\mathbf{a})$ is strictly less than $I(\lambda; \hat{\rho})$ except if all the $\hat{\rho}_j$ commute, and that this gap remains even asymptotically since

$$J_{\rho^{\otimes l}}(\mathbf{a}^{\otimes l}) = l \cdot J_\rho(\mathbf{a}).$$

Thus the mutual information at fixed measurement and fixed average state is in general strictly less than the amount of meaningful data produced by the measurement.

However, it follows from the results of [8,16,11] that there exists a measurement $\tilde{\mathbf{A}}$ acting on the tensor product $\mathcal{H}^{\otimes l}$ of the Hilbert space of the composite ensemble, such that its marginals are very close to \mathbf{a} , and such that the accessible information $J_{\rho^{\otimes l}}(\tilde{\mathbf{A}})$ equals $l \cdot I(\lambda; \hat{\rho})$ asymptotically.

We conjecture that the POVM \mathbf{A} constructed in theorem 2 has all the properties of $\tilde{\mathbf{A}}$ enumerated above. This would mean that the compressed version \mathbf{A} of the POVM $\mathbf{a}^{\otimes l}$ asymptotically closes the gap between mutual information and amount of data.

D. Open questions

There remain a number of open questions for future research of which we point out a few. The first three concern a better understanding of the conditions under which we get our result:

1. In the case where the ensemble on which the measurement is carried out is composed of mixed states, can one decrease further the amount of data produced by the measurement? The results proven in this paper use condition C3 in which only the average density matrix ρ of the states enters (through the definition of the marginal POVMs). However it is possible, if one uses the weaker conditions C0, C1, or C2 that the measurements can be further compressed.
2. Conversely, one could prove that further compression is impossible (theorem 3) using conditions C0, C1 or C2.
3. In the case of rank-one POVM the entropy defect in theorems 2 and 3 becomes the entropy of ρ , the number of outcomes of the compressed measurement is comparable to the dimension of the typical subspace of $\rho^{\otimes l}$. Since the interesting part of the construction is in the typical subspace we may ask whether one can achieve the bound of theorem 2 (or a slightly weaker one) by a *von Neumann* measurement. The methods used in the present paper and in [3] do not seem to yield this.

A final question concerns the tradeoff between fidelity and number of outcomes of \mathbf{A} . Here we studied only the extremal case where the fidelity should be arbitrarily close to the maximum, but comparison with rate distortion theory (see for example [17]) makes it plausible that by allowing a certain loss we can save even more in the output entropy. This is because on blocks the fidelity obeys the same form of rule as the typical distortion measures: it is the average over the block.

Several distortion criteria could be used, e.g.

$$F(\mathbf{A}) \geq F(\mathbf{a}) - d,$$

but many others seem natural, too.

A similar tradeoff may occur between the optimum compression rate and the parameter g , $\nu_l = \lfloor gl \rfloor$ in condition C5 (here we have treated only the case $g = 0+$).

We intend to pursue these questions in future work.

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